The Sphaleron in a Magnetic Field
and
Electroweak Baryogenesis

D. Comelli[1], D. Grasso[2,3], M. Pietroni[3] and A. Riotto[4]

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[1]INFN, Sezione di Ferrara,
Via Paradiso 12, I-44100 Ferrara, Italy
[2]Dipartimento di Fisica "G.Galilei",
Via F. Marzolo 8, I-35131 Padova, Italy
[3]INFN, Sezione di Padova,
Via F. Marzolo 8, I-35131 Padova, Italy
[4] CERN TH-Division,
CH-1211, Geneva 23, Switzerland

Abstract

The presence of a primordial magnetic field in the early universe affects the dynamic of the electroweak phase transition enhancing its strength. This effect may enlarge the window for electroweak baryogenesis in the minimal supersymmetric extension of the standard model or even resurrect the electroweak baryogenesis scenario in the standard model. We compute the sphaleron energy in the background of the magnetic field and show that, due to the sphaleron dipole moment, the barrier between topologically inequivalent vacua is lowered. Therefore, the preservation of the baryon asymmetry calls for a much stronger phase transition than required in the absence of a magnetic field. We show that this effect overwhelms the gain in the phase transition strength, and conclude that magnetic fields do not help electroweak baryogenesis.

1   Introduction

The *sphaleron* [1], is a static and unstable solution of the field equations of the electroweak model, corresponding to the top of the energy barrier between two topologically distinct

vacua. As was first noticed by 't Hooft [2], topological transitions are associated, via the axial anomaly, to the violation of baryon (B) and lepton (L) number, in the combination B+L. If thermal fluctuations in the early universe were strong enough to generate sphaleron configurations, then B+L was badly violated and any previously produced B+L–asymmetry would have been washed out. In any scenario of baryogenesis [3] it is then crucial to know at which epoch do the sphaleronic transitions fall out of thermal equilibrium. Generally this happens at temperatures below $\bar{T}$ such that [4]

$$\frac{E(\bar{T})}{\bar{T}} \geq A \ , \tag{1}$$

where $E(T)$ is the sphaleron energy at the temperature T and $A \simeq 35 - 45$, depending on the poorly known prefactor of the sphaleron rate. In the case of baryogenesis at the electroweak scale one requires the sphalerons to drop out of thermal equilibrium soon after the electroweak phase transition. It follows that the requirement $\bar{T} = T_c$, where T_c is the critical temperature, turns eq. (1) into a lower bound on the higgs vacuum expectation value (VEV),

$$\frac{v(T_c)}{T_c} \gtrsim 1 \ . \tag{2}$$

As a result of intense research in the recent years [5], it is by now agreed that the standard model (SM) does not have a phase transition strong enough as to fulfill eq. (2), whereas there is still some room left in the parameter space of the minimal supersymmetric standard model (MSSM) [3].

In refs. [6] an interesting observation was made, which could potentially revive baryogenesis even in the SM; if a magnetic field for the hypercharge $U(1)_Y$ was present for $T > T_c$, the electroweak phase transtion would have been delayed. The VEV soon after the transition would then have been larger and possibly able to fulfill eq. (2). The effect can be understood in analogy with Meissner effect, *i.e.* the expulsion of the magnetic field from superconductors. In our case, it is the Z–component of the hypercharge $U(1)_Y$ magnetic field which is expelled from the broken phase, and this process costs in terms of free energy. Then, compared to the case in which no magnetic field is present, it is less convenient for the system to go to the broken phase, and it has to wait until the effective potential becomes so deep as to compensate for the energy spent to expel the Z–component of the field.

The generation of magnetic fields before – or during [7] – the electroweak phase transition is predicted by several models, as is reviewed in ref. [8]

The perturbative estimates of refs. [6] gave very optimistic indications, concluding that eq. (2) could be fulfilled for higgs masses up to $m_H \simeq 100$ GeV if a hyper-magnetic field $B_Y \gtrsim 0.3\ T^2$ existed at the time of the electroweak phase transition. Lattice simulations [9] confirmed that the phase transition is strengthened by a magnetic field, but not enough as to give a viable baryogenesis scenario in the SM for $m_H \gtrsim 80$ GeV.

Even if these more pessimistic conclusions were correct, a magnetic field could still play a relevant role in models such as the MSSM, enlarging the portion of the parameter space available for baryogenesis. Analyzing further this scenario is therefore worth while.

The purpose of this letter is to point out and study an important effect which was not taken into account in previous discussions on the role of magnetic fields in electroweak baryogenesis, namely the large magnetic dipole moment of the sphaleron [1]. In the background of

a magnetic field the coupling with the dipole moment decreases the height of the sphaleron barrier, so that thermal fluctuations are more effective in producing topological transitions. In order to fulfill the freeze-out condition, eq. (1), it is then necessary to have larger VEV's than that of eq. (2), so that what has been gained in terms of strength of the phase transition may be lost by the fact that sphalerons have become lighter. We will show that this is indeed the case, at least for field values which can be studied reliably with our methods.

The paper is organized as follows. In the next section we shortly review the main equations which describe the sphaleron for finite values of the Weinberg angle. In section 3, an external magnetic field is introduced and we will determine its effect on the sphaleron energy and magnetic moment. Finally, in section 4, we decribe the consequence of our results on the electroweak baryogenesis scenario.

2 The Sphaleron dipole moment

In the limit of vanishing Weinberg angle, $\theta_w \to 0$, the sphaleron is a spherically symmetric, hedgehog-like configuration of $SU(2)$ gauge and Higgs fields [10]. No magnetic moment is present in this case. As θ_w is turned on the $U(1)$ field is excited and the spherical symmetry is reduced to an axial symmetry [1, 11, 12]. The most general Ansatz for the sphaleron at $\theta_w \neq 0$ was given in ref. [12], and requires seven independent scalar functions of the spherical coordinates r and θ. The θ-dependence is however very mild, and a very good approximation to the exact solution is obtained using the Ansatz by Klinkhamer and Laterveer [11], which requires four scalar functions of r only,

$$
\begin{aligned}
g' a_i \, dx^i &= (1 - f_0(\xi)) \, F_3 \,, \\
g W_i^a \sigma^a \, dx^i &= (1 - f(\xi))(F_1 \sigma^1 + F_2 \sigma^2) + (1 - f_3(\xi)) F_3 \sigma^3 \,, \\
\mathbf{\Phi} &= \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ h(\xi) \end{pmatrix} \,,
\end{aligned}
\tag{3}
$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge couplings, v is the higgs VEV such that $M_W = gv/2$, $M_h = \sqrt{2\lambda}v$, $\xi = gvr$, σ^a $(a = 1, 2, 3)$ are the Pauli matrices, and the F_a's are 1-forms defined as follows [11]

$$
\begin{aligned}
F_1 &= -2 \sin \phi \, d\theta - \sin 2\theta \cos \phi \, d\phi \,, \\
F_2 &= -2 \cos \phi \, d\theta + \sin 2\theta \sin \phi \, d\phi \,, \\
F_3 &= \ \ 2 \sin^2 \theta \, d\phi \,.
\end{aligned}
$$

The boundary conditions for the four scalar functions are

$$
f(\xi), \, f_3(\xi), \, h(\xi) \to 0 \,, \quad f_0(\xi) \to 1 \qquad\qquad \text{for } \ \xi \to 0
$$

$$
f(\xi), \, f_3(\xi), \, h(\xi), \, f_0(\xi) \to 1 \,, \qquad\qquad\qquad \text{for } \ \xi \to \infty \,. \tag{4}
$$

In ref. [12] it was shown that the above Ansatz can be recovered at the first order in a Legendre polynomials expansion of the exact sphaleron solution at $\theta_w \neq 0$, giving an excellent

approximation to the latter for θ_w of the order of the physical value $\theta_w = 0.5$ rads. Taking the $\theta_w \to 0$ limit the field equations give

$$f_3(\xi) \to f(\xi) , \qquad f_0(\xi) \to 1 \qquad \qquad (\text{for} \quad \theta_w \to 0)$$

for any value of ξ, thus reproducing the two-function Ansatz of the pure $SU(2)$-Higgs case [1]. The source of the hypercharge field a_i is the $O(\theta_w)$ current J_i

$$\partial_i f_{ij} = J_i$$

where

$$f_{ij} = \partial_i a_j - \partial_j a_i ,$$
$$J_i = -\frac{1}{2} i g' \left[\mathbf{\Phi}^\dagger D_i \mathbf{\Phi} - (D_i \mathbf{\Phi})^\dagger \mathbf{\Phi} \right] ,$$
$$D_i \mathbf{\Phi} = \partial_i \mathbf{\Phi} - \tfrac{1}{2} i g \sigma^a W_i^a \mathbf{\Phi} - \tfrac{1}{2} i g' a_i \mathbf{\Phi} .$$

The hypercharge field behaves asymptotically as a pure dipole

$$a_i \to \epsilon_{ijk} \frac{\mu_j \, x_k}{4\pi r^3} ,$$

with the dipole moment $\mu_j = \delta_{j3} \, \mu$. At the first order in θ_w, a_i can be neglected in J_i, which then takes the form

$$J_i^{(1)} = -\tfrac{1}{2} g' v^2 \frac{h^2(\xi)[1 - f(\xi)]}{r^2} \, \epsilon_{3ij} x_j , \tag{5}$$

where h and f are the solutions in the $\theta_w \to 0$ limit, giving for the dipole moment [1]

$$\mu^{(1)} = \frac{2\pi}{3} \frac{g'}{g^3 v} \int_0^\infty d\xi \xi^2 h^2(\xi)[1 - f(\xi)] . \tag{6}$$

The reader should note that the dipole moment is a true electromagnetic one because in the broken phase only the electromagnetic component of the hypercharge field survives at long distances.

3 Sphaleron in a magnetic field

If an external hypercharge field, a_i^c, is turned on, the energy functional is modified as

$$E = E_0 - E_{\text{dip}} , \tag{7}$$

with

$$E_0 = \int d^3x \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} f_{ij} f_{ij} + (D_i \mathbf{\Phi})^\dagger (D_i \mathbf{\Phi}) + V(\mathbf{\Phi}) \right] ,$$
$$F_{ij}^a = \partial_i W_j^a - \partial_j W_i^a + g \epsilon^{abc} W_i^b W_j^c ,$$
$$V(\mathbf{\Phi}) = \lambda \left(\mathbf{\Phi}^\dagger \mathbf{\Phi} - \tfrac{1}{2} v^2 \right)^2 ,$$

4

and

$$E_{\text{dip}} = \int d^3x J_i a_i^c = \tfrac{1}{2} \int d^3x f_{ij} f_{ij}^c \tag{8}$$

Besides the SM, our treatment will also cover the case of the MSSM in the parameter space in which one of the two neutral higgs directions decouples, *i.e.* when the mass of the pseudoscalar A^0 is much larger than the Z-boson mass and the low-energy potential reduces to a one-dimensional SM-like potential. Even though this is not the most favorable situation for the generation of the baryon asymmetry – since the latter turns out to be proportional to the variation of the ratio between the VEV's of the two neutral Higgs fields inside the bubble wall [13] – the strength of the MSSM phase transition is stronger. Our approximation is therefore well justified here, and eq. (7) can be thought as the energy for this effective theory even when thermal corrections are included and the VEV of the Higgs field v should be thought as temperature dependent.

We will consider a constant external hypermagnetic field B_Y^c directed along the x_3 axis, *i.e.*

$$a_i^c = -\epsilon_{3ij} \frac{B_Y^c}{2} x_j \,. \tag{9}$$

In the $\theta_w \to 0$ limit the sphaleron has no hypercharge contribution and then $E_{\text{dip}}^{(0)} = 0$. At $O(\theta_w)$, using (5) and (9) we get a simple dipole interaction

$$E_{\text{dip}}^{(1)} = \mu^{(1)} B_Y^c \,. \tag{10}$$

When the external field B_Y^c is zero, the $O(\theta_w)$ approximation is known to give values of the sphaleron energy with a better than percent accuracy [11, 12]. On the other hand, comparing $E_{\text{dip}}^{(1)}$ with E_0 – which is typically $O(M_W/g)$ – we realize that when $B_Y^c \neq 0$ the expansion parameter is effectively $\theta_w B_Y^c/(gv^2)$ so that, for large values of B_Y^c, higher orders in θ_w may become important. In order to assess the range of validity of the approximation (10) we need to go beyond the leading order in θ_w and look for a nonlinear B_Y^c–dependence of E.

First of all, we notice that the external field (9) does not break the axial symmetry, so we will use the same Ansatz as for the $B_Y^c = 0$ case, eq. (3). Moreover, the field equations are left unchanged by a constant B_Y^c ($\partial_i f_{ij}^c = 0$), *i.e.*

$$f'' + \frac{1-f}{4\,\xi^2} \left[8(f(f-2) + f_3 + f_3^2) + \xi^2 h^2 \right] = 0 \,,$$

$$f_3'' - \frac{2}{\xi^2} \left[3f_3 + f(f-2)(1+2f_3) \right] - \frac{h^2}{4}(f_3 - f_0) = 0 \,,$$

$$f_0'' + \frac{g_1^2}{4\,g^2} h^2(f_3 - f_0) + 2\frac{1-f_0}{\xi^2} = 0 \,,$$

$$h'' + \frac{2}{\xi}h' - \frac{2}{3\xi^2} h \left[2(f-1)^2 + (f_3 - f_0)^2 \right] - \frac{2\lambda}{3\,g^2}(h^2-1)h = 0 \,. \tag{11}$$

The only modification induced by B_Y^c resides in the boundary conditions since – as $r \to \infty$ – we now have

$$a_i = \cos\theta_W A_i - \sin\theta_W Z_i \to \cos\theta_W A_i^c = \cos^2\theta_w a_i^c$$

$$W_i^3 = \sin\theta_W A_i + \cos\theta_W Z_i \to \sin\theta_W A_i^c = \cos\theta_W \sin\theta_W a_i^c \,.$$

$$\tag{12}$$

The equalities on the right side of (12) have been obtained by requiring the continuity of the gauge fields at the boundary between the broken and the symmetric phase [6]. As a consequence of the new boundary conditions (12), eqs.(4) are turned into

$$f(\xi), h(\xi) \to 1, \qquad f_3(\xi), f_0(\xi) \to 1 - B_Y^c \sin 2\theta_w \frac{\xi^2}{8gv^2} \qquad \text{(for } \xi \to \infty)$$

wheras the boundary condition for $\xi \to 0$ are left unchanged. In order to keep the same form of the boundary conditions as for the case without the external field, it is convenient to define two new scalar functions, $g_0(\xi)$ and $g_3(\xi)$, as

$$g_{0(3)}(\xi) \equiv f_{0(3)}(\xi) + B_Y^c \sin 2\theta_w \frac{\xi^2}{8gv^2} \,. \tag{13}$$

The functions $f(\xi)$, $g_0(\xi)$, $g_3(\xi)$ and $h(\xi)$ have been determined by solving numerically eqs.(11). The result is plotted in fig.1 for $B_Y^c = 0$ (solid lines) and $B_Y^c = 0.19 \, v^2$ (dashed). The modifications between the two groups of curves is responsible for the nonlinear dependence of E on B_Y^c mentioned above.

In tab. 1 the values of E and of the effective electromagnetic dipole moment μ_{eff}, defined as

$$\mu_{eff}(B_Y^c) = \frac{\Delta E}{\cos \theta_W B_Y^c} \,, \tag{14}$$

whith $\Delta E \equiv E(B_Y^c) - E(B_Y^c = 0)$, are given for $M_h = M_W$. In the $B_Y^c \to 0$ limit, μ_{eff} tends to $\mu^{(1)}$ in eq. (6). From the variation of μ_{eff} in the considered B_Y^c–range we conclude that the corrections to the linear approximation

$$\Delta E \simeq \mu^{(1)} \cos \theta_W B_Y^c$$

are less than 5%.

For larger values of B_Y^c non-linear effects increase sharply. However, for such large magnetic fields the broken phase of the SM is believed to become unstable to the formation either of W-condensates [14] or of a mixed phase [9]. In such situations the sphaleron solution does not exist any more, so we will limit our considerations to safe values $B_Y^c \lesssim 0.4 \, T^2$.

B_Y^c/v^2	ε	m_{eff}
0	1.81	–
$6 \, 10^{-3}$	1.79	1.81
$3 \, 10^{-2}$	1.73	1.82
$6 \, 10^{-2}$	1.66	1.82
0.12	1.51	1.86
0.25	1.20	1.88

Table 1: Values for $\varepsilon = E \frac{g}{4\pi v}$ and $m_{eff} = \mu_{eff} \frac{\alpha_W M_W}{e}$ (see text) obtained for $M_h = M_W$.
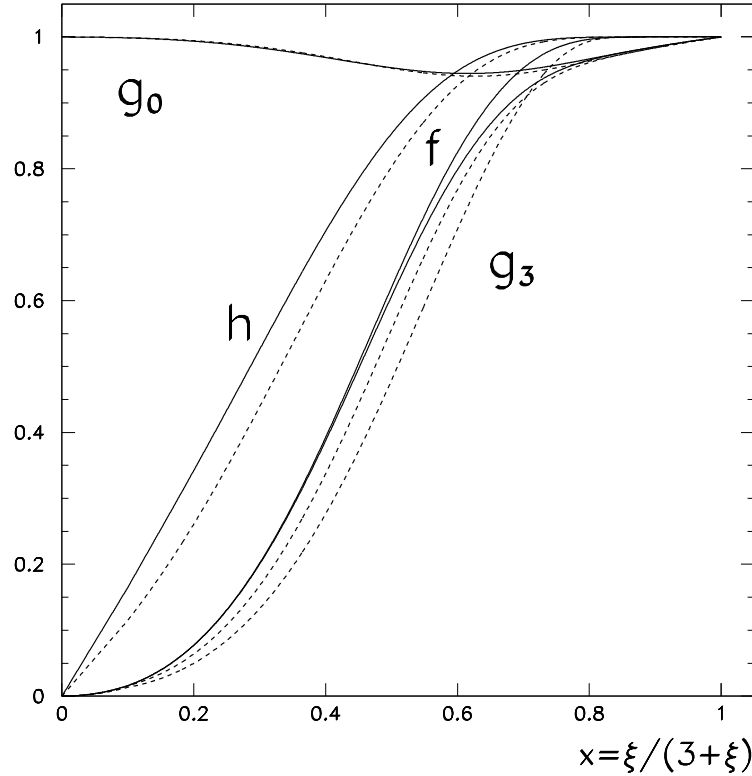
Figure 1: The scalar functions of the Ansatz (3) – redefined according to (13) – for $M_h = M_W$ and $B_Y^c = 0$ (solid), $0.2\ v^2$ (dashed).

4 Consequences for EW baryogenesis

We are now ready to discuss the effect of the sphaleron–magnetic field dipole interaction on the baryogenesis scenario discussed in refs. [6]. Let us first briefly recall its salient feature, namely the possibility of getting a stronger phase transition when the symmetric phase is permeated by a hypermagnetic field. Indeed, when this is the case, the pressure in the symmetric phase gets an extra contribution

$$P_u = -V(0) + \tfrac{1}{2} B_Y^{c\,2}\,.$$

In the broken phase only the electromagnetic component of the hypercharge field survives at long distances, the Z–component being screened as the Z–boson becomes massive, thus

$$P_b = -V(\Phi) + \tfrac{1}{2}\cos\theta_W^2 B_Y^{c\,2}\,.$$

The phase transition takes place at the temperature T_c where phase equilibration occurs,

$$V(0) - V(\phi) = \tfrac{1}{2}\sin^2\theta_w B_Y^{c\,2}\,, \qquad\qquad (\text{for}\quad T = T_c)\,. \qquad\qquad (15)$$

7

Considering the simple approximation to the SM potential,

$$V(\Phi) \simeq a(T^2 - T_b^2)\Phi^2 + \frac{\lambda}{4}\Phi^4 \,,$$

where the constant a needs not to be specified here, eq.(15) gives

$$\frac{v^2(T_c)}{T_c^2} = b\sqrt{\frac{2\sin^2\theta_w}{\lambda}} \,, \tag{16}$$

where $b \equiv B_Y^c/T_c^2$. If we fix $M_h = M_W$, we get $v(T_c)/T_c \gtrsim 1$ for $B_Y^c \gtrsim 0.33\, T_c^2$. The phase transition is then of the first order even in absence of a cubic term in the effective potential and the bound (2) is satisfied.

The conclusion that the sphaleron freeze-out condition (1) is satisfied and the baryon asymmetry preserved is however premature. Indeed, in an external magnetic field the relation between the VEV and the sphaleron energy is altered and eq. (2) does not imply (1) any more. We can understand it by considering the linear approximation to E,

$$E \simeq E(B_Y^c = 0) - \mu^{(1)}B_Y^c\cos\theta_W \equiv \frac{4\pi v}{g}\left(\varepsilon_0 - \frac{\sin 2\theta_w}{g}\frac{B_Y^c}{v^2}m^{(1)}\right) \tag{17}$$

where $m^{(1)}$ is the $O(\theta_W)$ dipole moment expressed in units of $e/\alpha_W M_W(T)$.

If $B_Y^c = 0$ the freeze-out condition $E/T > A$ translates straightforwardly into $v(T_c)/T_c > x_0 \equiv Ag/4\pi\varepsilon_0$, which – again for $M_h = M_W$ and $A = 35$ – gives (see tab. 1) $x_0 \simeq 1$. As B_Y^c is turned on, eq. (17) gives

$$\frac{v(T_c)}{T_c} > \frac{x_0}{2}\left(1 + \sqrt{1 + \frac{4\sin 2\theta_w\, m^{(1)}\, b}{g\,\varepsilon_0\, x_0^2}}\right) \,, \tag{18}$$

which gives $v/T_c \gtrsim 1.3$ for the same parameters as above and $b = 0.33$. Using (16) in (18) we see that there is no value of the field b for which the gain in terms of phase transition strength is enough to push the sphaleronic transitions out of thermal equilibrium.

The above estimate is confirmed by more accurate computations, as plotted in fig. 2, where our numerical values for E and the one-loop effective potential were used.

5 Summary

In this letter we have computed the sphaleron energy in the background of a constant magnetic field. Our main motivation was the investigation of a recently proposed scenario of electroweak baryogenesis, in which the phase transition is strengthened by a background hypermagnetic field. We have confirmed this effect, but pointed out that it is not enough as to preserve the baryon asymmetry produced before or during the phase transition. The main point is the dipole interaction between the sphaleron and the magnetic field which lowers the energy barrier between topologically inequivalent vacua, thus requiring a phase transition much stronger than that obtained by perturbative computations.
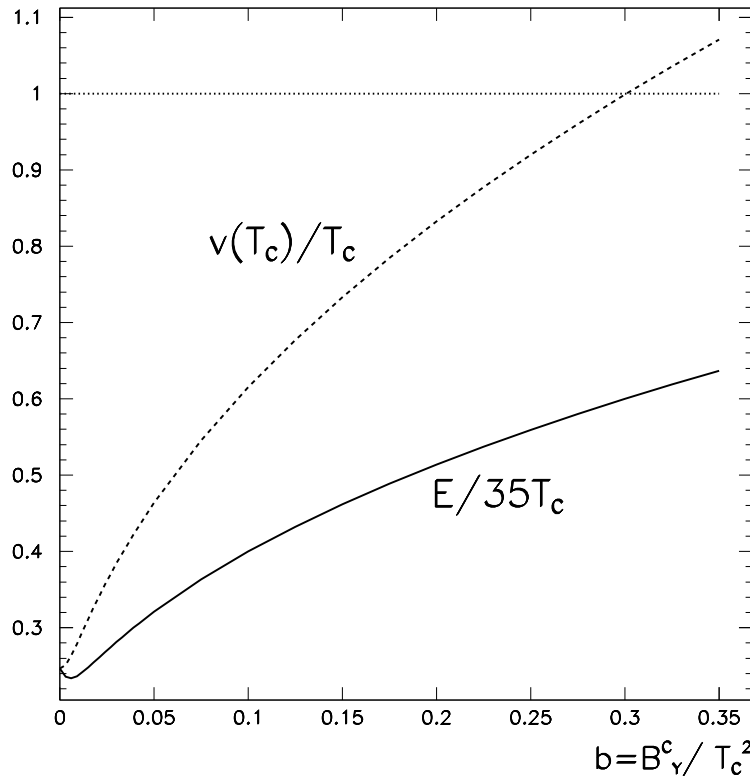
Figure 2: The VEV at the critical temperature, $v(T_c)$, and the sphaleron energy *vs.* the external magnetic field for $M_h = M_W$. Even if $v(T_c)/T_c \gtrsim 1$ the washout condition $E/T_c \gtrsim 35$ is far from being fulfilled.

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